Problem 1.7.

Prove the Ehrenfest's theorem.

$$\frac{d\langle p\rangle}{dt} = -\left\langle \frac{\partial V}{\partial x}\right\rangle$$

$$\langle p\rangle = \frac{\hbar}{i}\int_{-\infty}^{\infty} \psi \frac{\partial}{\partial x}\psi^* dx \,, \quad \frac{d\langle p\rangle}{dt} = \frac{\hbar}{i}\int_{-\infty}^{\infty} \frac{\partial}{\partial t}\left(\psi\frac{\partial}{\partial x}\psi^*\right)dx$$

Schrodinger's eqns.

$$-i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi \,, \quad -\frac{\hbar}{i}\frac{\partial\psi^*}{\partial t} = \frac{1}{2m}\left(\frac{\hbar}{-i}\frac{\partial}{\partial x}\right)^2\psi^* + V\psi^*$$

$$\frac{d\langle p\rangle}{dt} = \frac{\hbar}{i}\int_{-\infty}^{\infty}\left(\underset{f}{\frac{\partial\psi}{\partial t}}\underset{\partial g}{\frac{\partial\psi^*}{\partial x}}dx + \underset{f}{\psi}\frac{\partial}{\partial x}\underset{\partial g}{\frac{\partial\psi^*}{\partial t}}dx\right)$$

(integration by parts)

$$= -\frac{\hbar}{i}\int_{-\infty}^{\infty}\psi^*\frac{\partial}{\partial x}\frac{\partial\psi}{\partial t} + \cancel{\frac{\partial\psi}{\partial t}\psi^*}^{\,0} - \frac{\hbar}{i}\int_{-\infty}^{\infty}\frac{\partial\psi}{\partial x}\frac{\partial\psi^*}{\partial t}dx + \cancel{\psi\frac{\partial\psi^*}{\partial t}}^{\,0}$$

$$= -\frac{\hbar}{i}\int_{-\infty}^{\infty}\left[\psi^*\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial t}\right) + \frac{\partial\psi}{\partial x}\left(\frac{\partial\psi^*}{\partial t}\right)\right]dx$$

S's eqn

$$= \cancel{-\frac{\hbar}{i}} - \int_{-\infty}^{\infty}dx\, \psi^*\frac{\partial}{\partial x}\left[\frac{1}{2m}\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2\psi + V\psi\right]$$

$$+ \int_{-\infty}^{\infty}dx\,\frac{\partial\psi}{\partial x}\left[\frac{1}{2m}\left(\frac{\hbar}{-i}\frac{\partial}{\partial x}\right)^2\psi^* + V\psi^*\right].$$

$$= -\int_{-\infty}^{\infty}dx\,\psi^*\frac{\partial V\psi}{\partial x} - \int dx\,\underset{f}{\psi^*}\frac{1}{2m}\frac{\partial}{\partial x}\frac{\hbar}{i}\frac{\partial}{\partial x}\underset{g}{\frac{\hbar}{i}\frac{\partial\psi}{\partial x}}$$

$$+ \int dx\,\frac{\partial\psi}{\partial x}V\psi^* + \int dx\,\underset{f}{\frac{\partial\psi}{\partial x}}\frac{1}{2m}\frac{\hbar}{-i}\underset{g}{\frac{\partial}{\partial x}}\frac{\hbar}{-i}\frac{\partial\psi^*}{\partial x}$$

$$= -\int dx\,\psi^*\psi\frac{\partial V}{\partial x} - \int dx\,\cancel{V\psi^*\frac{\partial\psi}{\partial x}} + \int dx\,\cancel{V\psi^*\frac{\partial\psi}{\partial x}} + \text{Rest}$$